Mechanical Systems Laboratory: Lecture 8
Brief Review of Stability; PD Position Control of a Robot Arm

1. Brief Review of Stability

Stability refers to the concept of whether a system's performance "blows up" or converges to some value. What are some applications in which stability analysis is very important?

Airplanes, Spaceships, Active Suspensions for Cars, Surgery Robots

Three types of stability are:

stable, unstable, marginally stable (sustained oscillations)

The location of the poles of the transfer function determine the type of stability.
Why? Consider a second-order system:

$$H(s) = \frac{1}{s^2+as+b} = \frac{1}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} \qquad p_i = \text{"poles"}$$

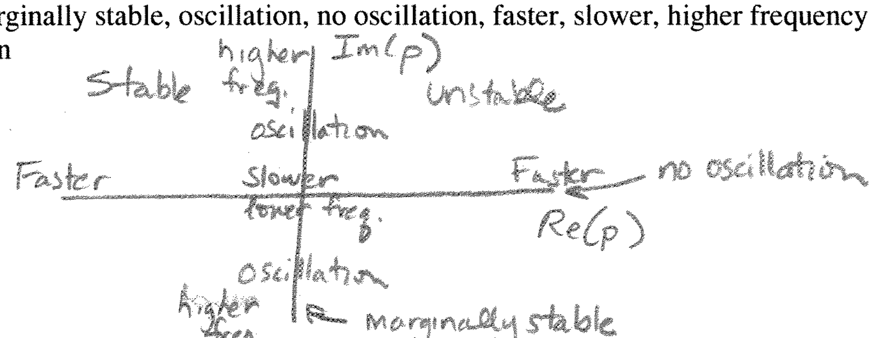
Remember, the inverse Laplace transform of the transfer function is the impulse response:

$$h(t) = Ae^{p_1 t} + Be^{p_2 t}$$

Case 1: $Re\{p_i\} < 0$	stable	exponential decay; if $\lvert Re\{p_2\}\rvert > \lvert Re\{p_1\}\rvert$ $e^{p_2 t}$ decays more quickly
Case 2: $Re\{p_i\} = 0$	marg. stable	if $Im\{p_i\} \neq 0 \Rightarrow$ sustained oscillation
Case 3: $Re\{p_i\} > 0$	unstable	exponential blow-up (w/ oscillation if $Im\{p_i\} \neq 0$)

So the location of the poles in the complex plane determines the type of response of the system.

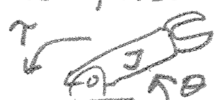
Exercise 1: Label the complex plane with the following words:
stable, unstable, marginally stable, oscillation, no oscillation, faster, slower, higher frequency oscillation, lower frequency oscillation



2. PD Position Control of a Robot Arm (P = proportional, D = Derivative)

Position control – most common industrial control system
Can you think of some applications? radar, robot arm, NC milling machine, manufacturing

Consider a one-joint robot arm:



Assume: 1) no friction or gravity; 2) we have a controller that can apply any torque that we want; 3) we can sense θ (for example, with a potentiometer)

Exercise 2: Design a proportional feedback controller to position the robot arm at $\theta=\theta_d$, find its transfer function, and analyze its stability

P-Control $\tau = -K_p(\theta - \theta_d)$

Robot Dynamics $\tau = J\ddot{\theta}$

Overall System Dynamics $J\ddot{\theta} = -K_p(\theta - \theta_d)$

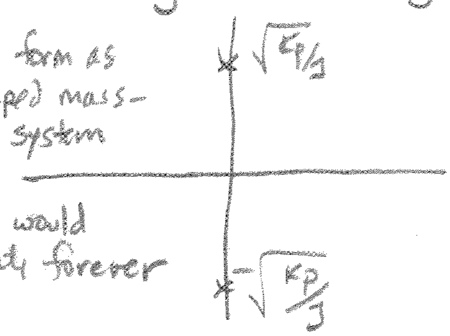
$J\ddot{\theta} + K_p\theta = K_p\theta_d$

Transfer Fun: $\dfrac{\Theta(s)}{\Theta_d(s)} = \dfrac{K_p}{Js^2 + K_p} = \dfrac{K_p/J}{s^2 + K_p/J}$

Poles: $s^2 = \dfrac{-K_p}{J} \Rightarrow s = \pm\sqrt{\dfrac{K_p}{J}}\, j$

Same form as undamped mass-spring system

Robot would oscillate forever



Exercise 3: Design a way to fix the problem. What kind of hardware would you need?

Add damping w/ control law: $\tau = -K_p(\theta - \theta_d) - K_v\dot{\theta}$] Proportional - Derivative Control

Two approaches to sensing angular velocity:
1) use a tachometer
2) differentiate position (e.g. using an op-amp circuit)

What are the dynamics and transfer function of the robot with the new controller?

$$J\ddot{\theta} = -K_p(\theta - \theta_d) - K_v\dot{\theta}$$
$$J\ddot{\theta} + K_v\dot{\theta} + K_p\theta = K_p\theta_d$$
$$\frac{\theta(s)}{\theta_d(s)} = H(s) = \frac{K_p}{Js^2 + K_v s + K_p}$$

We choose K_p + K_v when we design our controller!

How are the gains K_p and K_v related to the natural frequency and damping ratio?

$$H(s) = \frac{K_p/J}{s^2 + \frac{K_v}{J}s + \frac{K_p}{J}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n = \sqrt{\frac{K_p}{J}}$ K_p determines ω_n

$\zeta = \frac{K_v}{2\sqrt{K_p J}}$ After choosing K_p based on desired ω_n, can set damping ratio w/ K_v

What is the step response of the system?

$\zeta = 1$ $\theta(t) = 1 - \cos(\omega_n t)$

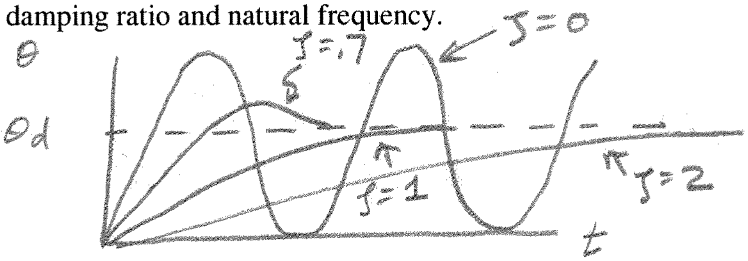
$0 < \zeta < 1$ $\theta(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

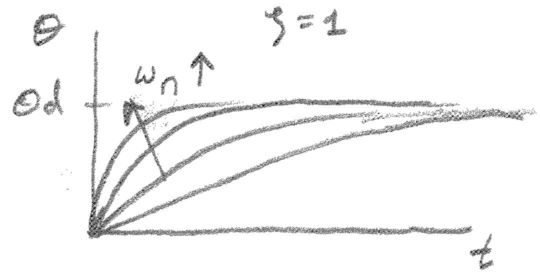
$\zeta = 1$ $\theta(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$

$\zeta > 1$ sum of two exponentials – see book; can neglect one exponential if $\zeta > 2$

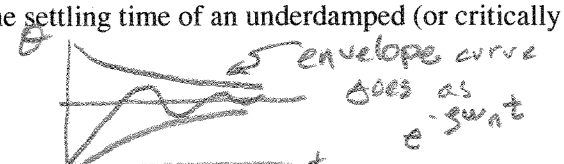
Note: the damping ratio determines whether the system oscillates (i.e. whether the poles have an imaginary part)

Exercise 4: Plot the step response of the system of the system would look like for different values of the damping ratio and natural frequency.





Notes: Overdamped systems are "sluggish"
Among systems responding without overshoot, critically damped systems exhibit the fastest response.
Underdamped systems with $0.5 < \zeta < .8$ get close to final value more rapidly than critically damped or overdamped systems.
The settling time of an underdamped (or critically damped) system is:



$e^{-t/\tau}$ is w/in 5% of final value after 3τ secs

$t_s\Big|_{5\%\ \text{criterion}} \approx 3\tau = \frac{3}{\zeta\omega_n}$ (minimized at $\zeta = .68$)

Exercise 5: Given a one-joint robot arm (no friction, no gravity) with $J = 1\ \text{kgm}^2$. Design a PD position controller such that the robot finishes 95% of a commanded step-function movement in .5 seconds, with no overshoot.

$t_s = \frac{3}{\zeta\omega_n} = .5$ $\zeta = 1$ (no overshoot) $\Rightarrow \omega_n = \frac{3}{.5} = 6.0\ \text{rad/sec}$

$\omega_n = \sqrt{\frac{K_p}{J}} \Rightarrow K_p = \omega_n^2 J = 36\ \frac{\text{kg}\,\text{m}^2}{\text{s}^2}$ $\zeta = \frac{K_v}{2\omega_n J} \Rightarrow K_v = 2\omega_n J\zeta = 12\ \frac{\text{kg}\,\text{m}^2}{\text{sec}}$